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# Ansatz of Hans Bethe for a two-dimensional lattice Bose gas\*

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## Abstract

The method of  $q$ -oscillator lattices, proposed recently in Bazhanov and Sergeev 2005 (*Preprint* hep-th/0509181), provides the tool for a construction of various integrable models of quantum mechanics in  $(2 + 1)$ -dimensional spacetime. In contrast to any one-dimensional quantum chain, its two-dimensional generalizations—quantum lattices—admit different geometrical structures. In this paper, we consider the  $q$ -oscillator model on a special lattice. The model may be interpreted as a two-dimensional lattice Bose gas. The most remarkable feature of the model is that it allows the coordinate Bethe ansatz: the  $p$ -particles' wavefunction is the sum of plane waves. Consistency conditions is the set of  $2p$  equations for  $p$  one-particle wave vectors. These 'Bethe ansatz' equations are the main result of this paper.

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## 1. Introduction

Integrable models of quantum mechanics in discrete spacetime describe a system of interacting quantum observables (spin operators, oscillators, etc) situated in sites of a one-dimensional chain (the models in  $(1 + 1)$ -dimensional spacetime) or in the vertices of a two-dimensional lattice (models in  $(2 + 1)$ -dimensional spacetime). The subject of this paper is the second variant—the quantum integrable lattices.

Formulation of a *realistic* integrable model of quantum mechanics in  $(2 + 1)$ -dimensional spacetime was a long-standing problem in the theory of integrable systems. Examples of higher dimensional integrable systems are known, but the integrability seemed to be a very high price paid to the detriment of a physical interpretation.

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Both known classes of integrable quantum lattices may be specified by the definition of the algebra of observables. In contrast to the quantum chains, where the algebra of observables may be, e.g., an evaluation representation of any quantum group, in the case of quantum lattices the algebra of observables may be either the local<sup>1</sup> Weyl algebra [1] or the local  $q$ -oscillator algebra [2]. Other examples of integrable quantum lattices are not known.

When the algebra of observables (and its representation) is fixed, there exists an almost unique way to produce the key notion of the integrability—a complete set of commutative operators. For the spin chains, it is the way of Lax operators and transfer matrices with, e.g., fundamental representations in the auxiliary space. In both classes of quantum lattices, the integrals of motion may be produced by a decomposition of a determinant of a certain big operator-valued matrix with respect to two spectral parameters [3, 4]. Therefore, the integrals of motion rather have a combinatorial nature. Both classes of quantum lattices allow a definition of another kind of operators: the layer-to-layer transfer matrices, related to intertwining relations, tetrahedron equations, etc. Such representation-dependent layer-to-layer transfer matrices are the kind of discrete-time evolution operators for the quantum lattices, their structure is much more complicated, but since their integrals of motion are known, it is not necessary to consider them at the first step.

The concept of quantum lattices implies a certain feature which is not applicable to the quantum chains. One may vary not only a size of the lattice, but its geometry and topology as well. This is the reason why we discuss the *classes* of models. The methods of Weyl algebra and  $q$ -oscillator algebra are the frameworks for construction of various quantum lattices.

Mainly the simplest geometry was considered before—the square lattices with periodical boundary conditions<sup>2</sup>. The simple square lattice with the sizes  $n \times m$  may be regarded as the length- $m$  chain of its length- $n$  lines, and therefore such quantum lattice is effectively a quantum chain. In particular, the quantum lattice with the Weyl algebra at a root of unity corresponds to auxiliary transfer matrices [1] of the  $\mathcal{U}_q(\widehat{\mathfrak{gl}}_n)$  generalized chiral Potts model [6, 7]. The  $q$ -oscillator quantum lattice corresponds to the auxiliary transfer matrices for reducible oscillator representation of  $\mathcal{U}_q(\widehat{\mathfrak{gl}}_n)$  [2]. From the point of view of quantum mechanics in  $(2+1)$ -dimensional spacetime, the square lattices provide a non-realistic models since the translation operators do not belong to the set of integrals of motion and therefore momenta of eigenstates cannot be defined. In the quantum chain interpretation, it means that  $U(1)$  charges are the variables, and the chain is homogeneous only on subspaces where all  $U(1)$  charges are the same in all sites. Otherwise, the translation invariance is lost.

The issue is evident. One has to consider a lattice which is not a chain of its lines. We define such lattice in the next sections. The framework of  $q$ -oscillator model on our special lattice immediately produces a kind of two-dimensional lattice Bose gas with physical dynamics. The Fock vacuum is the natural reference state, the total  $q$ -oscillators occupation number—the number of bosons—is conserved, and the  $p$ -particles' wavefunction is the superposition of one-particle plane waves. Each plane wave is characterized by two components of its momentum, and the consistency conditions give  $2p$  equations for  $p$  two-components momenta. These consistency conditions look much more complicated than the Bethe ansatz equations for a quantum chain, we do not investigate them in details here. The aim of this paper is just to present the method, the model and the 'Bethe ansatz' equations.

<sup>1</sup> The term 'local' means that the quantum observables in different vertices commute, as well as it is in the quantum chains case.

<sup>2</sup> Examples of Weyl algebra non-square lattices were mentioned in [5]. In particular, the Weyl algebra framework on the different shapes of the lattices produces the relativistic Toda chain as well as the quantum discrete Liouville model.

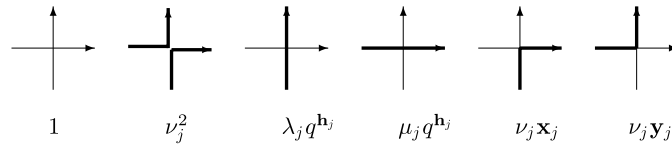


Figure 1. The ‘weights’  $f_j$  of  $j$ th vertex,  $j = 1, \dots, \Delta$ .

### 2. Framework of $q$ -oscillator algebra

We start with the formulation of a generic  $q$ -oscillator lattice.

Let  $\mathcal{L}$  be a lattice formed by a number of directed lines on a torus. Pairwise intersections of the lines are the vertices of the lattice. Let the vertices be enumerated in same way, for instance by the integer index  $j = 1, 2, 3, \dots, \Delta$ , where  $\Delta$  is the number of vertices.

The  $q$ -oscillator generators  $\mathbf{x}_j, \mathbf{y}_j$  and  $\mathbf{h}_j$  are assigned to vertex  $j$ . The oscillators for the different vertices commute. Locally, we define the  $q$ -oscillator algebra by

$$\mathbf{x}_j \mathbf{y}_j = 1 - q^{2+2\mathbf{h}_j}, \quad \mathbf{y}_j \mathbf{x}_j = 1 - q^{2\mathbf{h}_j}, \quad \mathbf{x}_j q^{\mathbf{h}_j} = q^{\mathbf{h}_j+1} \mathbf{x}_j, \quad \mathbf{y}_j q^{\mathbf{h}_j} = q^{\mathbf{h}_j-1} \mathbf{y}_j. \tag{1}$$

Consider next the system of non-self-intersecting paths along the edges of the lattice. The lines of the lattice are directed, and we demand that paths must follow the orientation of the edges. The paths may go through a single vertex in one of six variants shown in figure 1. The vertex ‘weight’  $f_j$  is associated with each variant of bypassing as it is shown in figure 1. Coefficients  $\lambda_j, \mu_j$  and  $\nu_j$  are extra vertex  $\mathbb{C}$ -valued parameters, they are related by  $\nu_j^2 \equiv -q^{-1} \lambda_j \mu_j$ . In general, they are free up to a single condition: if two vertices  $j$  and  $j'$  are formed by the intersections of the same lines, their  $\mathbb{C}$ -valued parameter must be the same [4].

For any path  $P$  let

$$\mathbf{t}_P = \prod_{\text{along } P} f_j. \tag{2}$$

Note that, if a path does not touch a vertex, then it vertex’s contribution to  $\mathbf{t}_P$  is just the unity according to the leftmost variant of figure 1. Recall in addition, the  $q$ -oscillator algebra is local, elements of different vertices commute. Therefore, the notion of the product along the path in (2) is well defined.

Let  $A$  and  $B$  be the two basic homotopy cycles of the torus. Any path has a certain homotopy class,  $P \sim nA + mB$ . Let

$$\mathbf{t}_{n,m} = \sum_{P: P \sim nA+mB} \mathbf{t}_P. \tag{3}$$

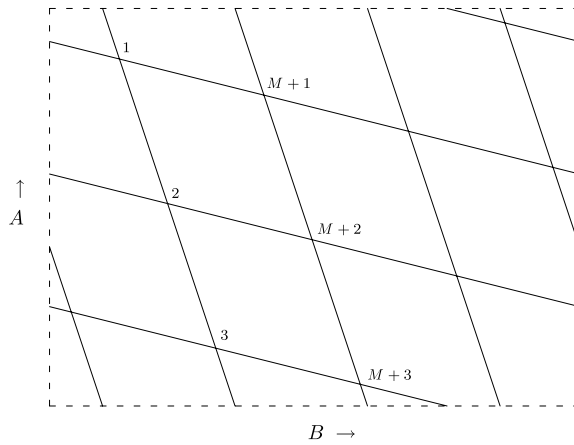
Here, the sum is taken over all possible paths of homotopy class  $nA + mB$ . Formula (3) ends the formulation of  $q$ -oscillator lattice’s prescription. The point is that for any lattice  $\mathcal{L}$ , formed by directed lines on the torus, the operators  $\mathbf{t}_{n,m}$  form the commutative set

$$\mathbf{t}_{n,m} \mathbf{t}_{n',m'} = \mathbf{t}_{n',m'} \mathbf{t}_{n,m} \quad \forall n, m, n', m'. \tag{4}$$

Moreover, if there are no lines of trivial homotopy class in  $\mathcal{L}$ , the set of integrals of motion is complete.

Integrals of motion may be combined into a polynomial of two variables,

$$\mathbf{T}(u, v) = \sum_{n,m} u^n v^m \mathbf{t}_{n,m}. \tag{5}$$



**Figure 2.** The lattice formed by the lines  $\alpha \sim 4A - B$  and  $\beta \sim 3B - A$ .

Operator  $\mathbf{T}$  has the structure of the layer-to-layer transfer matrix, equations (4) are equivalent to the commutativity of transfer matrices with different spectral parameters  $u, v$ . The commutativity may be proven for simple lattices with the help of tetrahedron equation [2, 8, 9].

### 3. The lattice

Turn now to the definition of our special lattice. It is formed by the intersections of only two lines  $\alpha$  and  $\beta$ . With respect to the basic cycles  $A$  and  $B$  of the torus, the lines have the classes

$$\alpha \sim NA - B, \quad \beta \sim MB - A. \tag{6}$$

Example of such lattice is given in figure 2. There the opposite dashed borders are identified (dashed lines are the cuts of the torus). The line  $\alpha$  is directed up (along the  $A$ -cycle), the line  $\beta$  is directed to the right (along the  $B$ -cycle). The lines are intersecting in

$$\Delta = NM - 1 \tag{7}$$

points.

We enumerate the vertices by number  $j, j \in \mathbb{Z}_\Delta$ . The numeration is successive along the reverse direction of  $\alpha$ -line, see figure 2. The shift  $j \rightarrow j - 1$  corresponds to a one-step translation along the  $\alpha$ -line, whereas the shift  $j \rightarrow j + M$  corresponds to the one-step translation along the  $\beta$ -line.

Now the lattice is formulated, and combinatorial rules of figure 1 may be applied. Note that, since all the vertices of the lattice are formed by the same lines, all their  $\mathbb{C}$ -valued parameters are the same, and one may put

$$\lambda = \mu = 1, \quad v^2 = -q^{-1}. \tag{8}$$

Generating function (5) has the structure

$$\mathbf{T}(u, v) = u^N v^{-1} q^{\mathcal{N}} + v^M u^{-1} q^{\mathcal{N}} + \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u^n v^m \mathbf{t}_{n,m}, \tag{9}$$

where  $\mathcal{N}$  is the total occupation number

$$\mathcal{N} = \sum_j \mathbf{h}_j. \tag{10}$$

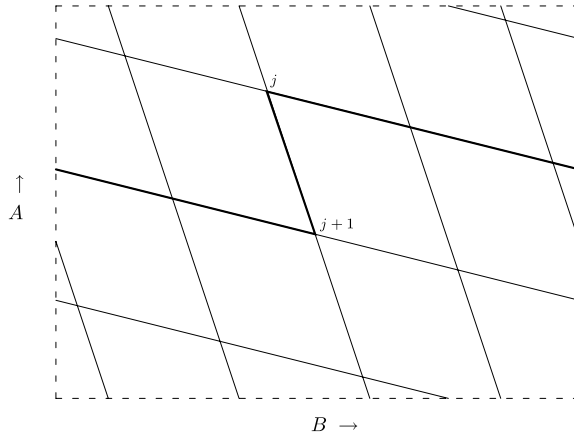


Figure 3. Path of the class  $B$ —an element of  $\mathbf{t}_{0,1}$ .

Elements  $q^N$  in (9) are the values of both  $\mathbf{t}_{N,-1}$  and  $\mathbf{t}_{-1,M}$ , combinatorially they are the paths along all  $\alpha$ - or  $\beta$ -lines (the third and the fourth variants of figure 1 for all vertices). Elements  $\mathbf{t}_{0,0}$  and  $\mathbf{t}_{N-1,M-1}$  correspond to the empty path and the complete path (the first and the second variants of figure 1 for all vertices) correspondingly, their values are

$$\mathbf{t}_{0,0} = 1, \quad \mathbf{t}_{N-1,M-1} = (-q^{-1})^\Delta. \tag{11}$$

In what follows, we will consider the elements  $\mathbf{t}_{0,1}$  and  $\mathbf{t}_{1,0}$  given by

$$\begin{aligned} \mathbf{t}_{1,0} &= -q^{-1} \sum_j \mathbf{x}_j \mathbf{y}_{j+M} q^{\mathbf{h}_{j+1} + \mathbf{h}_{j+2} + \dots + \mathbf{h}_{j+M-1}}, \\ \mathbf{t}_{0,1} &= -q^{-1} \sum_j \mathbf{x}_j \mathbf{y}_{j+1} q^{\mathbf{h}_{j+M} + \mathbf{h}_{j+2M} + \dots + \mathbf{h}_{j+(N-1)M}}. \end{aligned} \tag{12}$$

Combinatorial summand of  $\mathbf{t}_{0,1}$  is shown in figure 3.

#### 4. Eigenstates

The total occupation number is the conserving quantity, and therefore we may construct the eigenstates of the model step-by-step just increasing the number of bosons for the Fock space representation of the  $q$ -oscillator algebra. The Fock vacuum  $|0\rangle$  is defined by

$$\mathbf{x}_j |0\rangle = 0, \quad \mathbf{h}_j |0\rangle = 0 \quad \forall j \in \mathbb{Z}_\Delta. \tag{13}$$

The Fock vacuum is evidently the eigenstate of the model, eigenvalues of all  $\mathbf{t}_{n,m}$  are zeros (except  $\mathbf{t}_{0,0}$ ,  $\mathbf{t}_{N-1,M-1}$  and  $q^N$ ).

In what follows, we concentrate on the diagonalization of  $\mathbf{t}_{0,1}$ . Its eigenstates are defined uniquely. They will be evidently the eigenstates of  $\mathbf{t}_{1,0}$  and, due to the uniqueness, they must be the eigenstates of all the other integrals of motion. We will use below the normalized form of (12):

$$\tau_\alpha = -\frac{q}{1-q^2} \mathbf{t}_{0,1}, \quad \tau_\beta = -\frac{q}{1-q^2} \mathbf{t}_{1,0}. \tag{14}$$

#### 4.1. One boson state

Consider the states with the single boson,  $\mathcal{N} = 1$ . Let

$$|\phi_j\rangle = \mathbf{y}_j|0\rangle \quad (15)$$

be the basis of one-boson states. On this subspace, both operators (14) are the translation operators:

$$\tau_\alpha|\phi_j\rangle = |\phi_{j+1}\rangle, \quad \tau_\beta|\phi_j\rangle = |\phi_{j+M}\rangle. \quad (16)$$

Therefore,

$$|\Psi\rangle = \sum_j \omega^j |\phi_j\rangle, \quad \omega^\Delta = 1 \quad (17)$$

is the eigenstate of the model, the eigenvalues of (14) are

$$\tau_\alpha|\Psi\rangle = |\Psi\rangle\omega^{-1}, \quad \tau_\beta|\Psi\rangle = |\Psi\rangle\omega^{-M}. \quad (18)$$

Since  $\omega$ ,  $\omega^\Delta = 1$  may take  $\Delta$  different values, the set of (17) is complete on the subspace  $\mathcal{N} = 1$ .

Formally, the spectrum of both translation operators (18) is defined by one parameter  $\omega$ . Note that eigenvalues  $\omega^{-1}$  and  $\omega^{-M}$  are dual in the sense  $(\omega^{-M})^N = \omega^{-1}$ . When  $N, M \rightarrow \infty$ , one may talk about two independent components of momentum. Namely, let

$$k = Nk_\alpha + k_\beta, \quad (19)$$

where  $k_\alpha$  and  $k_\beta$  are relatively small. Then, when  $N, M \rightarrow \infty$ ,

$$\omega^{-1} = e^{2\pi ik/\Delta} \rightarrow e^{2\pi ik_\alpha/M}, \quad \omega^{-M} = e^{2\pi iMk/\Delta} \rightarrow e^{2\pi ik_\beta/N}, \quad (20)$$

i.e., in the thermodynamical limit  $2\pi k_\alpha/M$  and  $2\pi k_\beta/N$  play the roles of independent components of the momentum.

#### 4.2. Two-boson state

Turn next to  $\mathcal{N} = 2$ . Define

$$c_{j,k} = \delta_{j,k+M} + \delta_{j,k+2M} + \dots + \delta_{j,k+(N-1)M}. \quad (21)$$

Then,

$$\begin{aligned} \tau_\alpha \cdot \mathbf{y}_j \mathbf{y}_k |0\rangle &= (q^{c_{k,j}} \mathbf{y}_{j+1} \mathbf{y}_k + q^{c_{j,k}} \mathbf{y}_j \mathbf{y}_{k+1}) |0\rangle, \quad j \neq k, \\ \tau_\alpha \cdot \mathbf{y}_k^2 |0\rangle &= (1 + q^2) \mathbf{y}_{k+1} \mathbf{y}_k |0\rangle. \end{aligned} \quad (22)$$

Consider a two-particles' state with zero momentum of the mass centre:

$$|\Psi\rangle = \sum_j \left( \Psi_0 (1 + q^2)^{-1} \mathbf{y}_j^2 |0\rangle + \sum_{k>0} \Psi_k \mathbf{y}_j \mathbf{y}_{j+k} |0\rangle \right) \quad (23)$$

Eigenvalue equation  $\tau_\alpha|\Psi\rangle = |\Psi\rangle\tau_\alpha$  reads in components

$$\begin{aligned} \tau_\alpha \Psi_0 &= (1 + q^2) \Psi_1, \\ \tau_\alpha \Psi_{nM-1} &= q \Psi_{nM} + \Psi_{nM-2}, \\ \tau_\alpha \Psi_{nM} &= \Psi_{nM+1} + \Psi_{nM-1}, \\ \text{in all other cases: } \tau_\alpha \Psi_k &= \Psi_{k-1} + \Psi_{k+1}. \end{aligned} \quad (24)$$

Periodicity condition is simply

$$\Psi_k = \Psi_{\Delta-k}. \quad (25)$$

Define the double index:  $j = Mn + m \rightarrow j = (n, m)$ ,

$$\Psi_{Mn+m} = \Psi_{(n,m)}, \quad 0 \leq m < M, \quad 0 \leq n < N. \quad (26)$$

Equations (24) have in the general position the form  $\tau_\alpha \Psi_{(n,m)} = \Psi_{(n,m-1)} + \Psi_{(n,m+1)}$ . Therefore,

$$\Psi_{(n,m)} = P_n \omega^m + Q_n \omega^{-m}, \quad \tau_\alpha = \omega + \omega^{-1}. \quad (27)$$

Initial conditions (the first of (24)) give

$$\frac{Q_0}{P_0} = \frac{1 - q^2 \omega^2}{q^2 - \omega^2}. \quad (28)$$

The second and third relations of (24) give

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \mathcal{M} \cdot \begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \omega^M \frac{q^2 - \omega^2}{q(1 - \omega^2)} & \omega^{-M} \frac{\omega^2(q^2 - 1)}{q(1 - \omega^2)} \\ \omega^M \frac{1 - q^2}{q(1 - \omega^2)} & \omega^{-M} \frac{1 - q^2 \omega^2}{q(1 - \omega^2)} \end{pmatrix}. \quad (29)$$

Let  $\Omega$  and  $\Omega^{-1}$  be the eigenvalues of  $\mathcal{M}$ . Then, one may choose the basis of eigenvectors of  $\mathcal{M}$  such that

$$P_n = A_{++} \Omega^n + A_{-+} \Omega^{-n}, \quad Q_n = A_{+-} \Omega^n + A_{--} \Omega^{-n}. \quad (30)$$

The boundary condition  $\Psi_k = \Psi_{\Delta-k}$  reads in double indices  $\Psi_{(N-1,m)} = \Psi_{(0,M-1-m)}$ , i.e.,

$$\begin{pmatrix} P_{N-1} \\ Q_{N-1} \end{pmatrix} = \mathcal{M}^{N-1} \cdot \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} = \begin{pmatrix} \omega^{1-M} Q_0 \\ \omega^{M-1} P_0 \end{pmatrix}. \quad (31)$$

The following pair of equations summarizes all the calculations:

$$\begin{aligned} \Omega + \Omega^{-1} &= \omega^M \frac{q^2 - \omega^2}{q(1 - \omega^2)} + \omega^{-M} \frac{1 - q^2 \omega^2}{q(1 - \omega^2)}, \\ \omega + \omega^{-1} &= \Omega^N \frac{q^2 - \Omega^2}{q(1 - \Omega^2)} + \Omega^{-N} \frac{1 - q^2 \Omega^2}{q(1 - \Omega^2)}. \end{aligned} \quad (32)$$

Here, the first equation is the definition of  $\Omega$  (it is the characteristic polynomial of  $\mathcal{M}$ ). The second equation comes from (31) after excluding of  $P_0, Q_0$  via (28). The final answer is the Bethe ansatz (two-particles' wavefunction is the superposition of one-particle plane waves)

$$\Psi_{(n,m)} = A_{++} \Omega^n \omega^m + A_{-+} \Omega^{-n} \omega^m + A_{+-} \Omega^n \omega^{-m} + A_{--} \Omega^{-n} \omega^{-m}. \quad (33)$$

Parameters  $A_{\pm,\pm}$ , related to the eigenvectors of  $\mathcal{M}$ , are defined by

$$\Omega^{N-1} A_{+-} = \omega^{M-1} A_{-+}, \quad A_{--} = \Omega^{N-1} \omega^{M-1} A_{++} \quad (34)$$

and

$$\frac{A_{+-} + A_{--}}{A_{++} + A_{-+}} = \frac{1 - q^2 \omega^2}{q^2 - \omega^2}, \quad \frac{A_{-+} + A_{--}}{A_{++} + A_{+-}} = \frac{1 - q^2 \Omega^2}{q^2 - \Omega^2}. \quad (35)$$

The state (33) has the evident  $N \leftrightarrow M$  symmetry:

$$\tau_\alpha |\Psi\rangle = (\omega + \omega^{-1}) |\Psi\rangle, \quad \tau_\beta |\Psi\rangle = (\Omega + \Omega^{-1}) |\Psi\rangle. \quad (36)$$

From the point of view of  $\tau_\beta$ , the second relation of (32) is the characteristic polynomial, while the first one comes from the  $\beta$ -boundary conditions.



4.3. *p*-boson state

Turn finally to the state with  $\mathcal{N} = p$  bosons. Actually, all the results of this section were obtained explicitly for  $p = 3, 4$  and then conjectured for arbitrary  $p$ .

The wavefunction  $\Psi_{j_1, j_2, \dots, j_p}$  of the eigenstate

$$|\Psi\rangle = \sum_{j_1 \leq j_2 \leq \dots \leq j_p} \Psi_{j_1, j_2, \dots, j_p} \mathbf{y}_{j_1} \mathbf{y}_{j_2} \cdots \mathbf{y}_{j_p} |0\rangle \tag{37}$$

is the superposition of the plane waves. In the generic point (no coincidence in  $j_1, j_2, \dots, j_p$ ), the wavefunction is

$$\Psi_{j_1, j_2, \dots, j_p} = \sum_{\sigma, \sigma'} A_{\sigma', \sigma} \Omega_{\sigma'_1}^{n_1} \Omega_{\sigma'_2}^{n_2} \cdots \Omega_{\sigma'_p}^{n_p} \omega_{\sigma_1}^{m_1} \omega_{\sigma_2}^{m_2} \cdots \omega_{\sigma_p}^{m_p} \tag{38}$$

where  $\sigma$  and  $\sigma'$  are independent permutations of the set  $(1, 2, \dots, p)$  and  $n_a, m_a$  are related with  $j_a$  by

$$j_a = Mn_a + m_a, \quad 0 \leq m_a < M, \quad 0 \leq n_a < N, \tag{39}$$

cf (26). Two sets of exponential momenta

$$\omega_a = e^{ik_a}, \quad \Omega_a = e^{ik'_a}, \quad a = 1, 2, \dots, p \tag{40}$$

are solutions of two sets of equations. To write these equation, we need some extra notation. Firstly, let

$$G_{a,b}(\{\omega\}) = \frac{q^{-1}\omega_b - q\omega_a}{\omega_b - \omega_a}. \tag{41}$$

Let  $I_k$  be a length- $k$  subsequence of  $(1, 2, \dots, p)$ ,  $I_{p-k}$  be the compliment subsequence such that

$$I_k \cup I_{p-k} = (1, 2, \dots, p). \tag{42}$$

Let then

$$P_k[\{\omega, \omega^M\}] = \sum_{I_k} \left( \prod_{a \in I_k} \omega_a^M \prod_{b \in I_{p-k}} G_{b,a}(\{\omega\}) \right), \tag{43}$$

where the sum is taken over all possible subsequences of the length  $k$ . For example, for  $p = 3$

$$\begin{aligned} P_1[\{\omega, \omega^M\}] &= \omega_1^M G_{21} G_{31} + \omega_2^M G_{12} G_{32} + \omega_3^M G_{13} G_{23}, \\ P_2[\{\omega, \omega^M\}] &= \omega_1^M \omega_2^M G_{32} G_{31} + \omega_1^M \omega_3^M G_{21} G_{23} + \omega_2^M \omega_3^M G_{12} G_{13}, \\ P_3[\{\omega, \omega^M\}] &= \omega_1^M \omega_2^M \omega_3^M. \end{aligned} \tag{44}$$

The final step: let

$$\mathfrak{P}(\Omega|\{\omega, \omega^M\}) = \sum_{k=0}^p (-)^k \Omega^{p-k} P_k[\{\omega, \omega^M\}] = \Omega^p - \Omega^{p-1} P_1 + \Omega^{p-2} P_2 + \cdots. \tag{45}$$

Then the consistency conditions for the ansatz (38) read

$$\begin{cases} \mathfrak{P}(\Omega_a|\{\omega, \omega^M\}) = 0, \\ \mathfrak{P}(\omega_a|\{\Omega, \Omega^N\}) = 0, \end{cases} \quad \forall a = 1, 2, \dots, p. \tag{46}$$

These are the Bethe ansatz equations for our model.

The values of  $\Psi_{j_1, j_2, \dots, j_p}$  in the case when some of  $j_a$  coincide, as well as the values of  $(p!)^2$  amplitudes  $A_{\sigma, \sigma'}$ , can be defined uniquely.

The state (38) provides the eigenvalues of (14)

$$\tau_\alpha|\Psi\rangle = |\Psi\rangle \left( \sum_{a=1}^p \omega_a^{-1} \right), \quad \tau_\beta|\Psi\rangle = |\Psi\rangle \left( \sum_{a=1}^p \Omega_a^{-1} \right). \tag{47}$$

One may introduce the Hamiltonian of the system,

$$\mathbf{H} = 2\mathcal{N} - \frac{1}{2}(\tau_\alpha + \tau_\alpha^\dagger + \tau_\beta + \tau_\beta^\dagger), \tag{48}$$

where  $\mathcal{N}$  is the total occupation number (10). Spectrum of  $\mathbf{H}$  is

$$\mathbf{H}|\Psi\rangle = |\Psi\rangle \left( \sum_{a=1}^p \varepsilon_a \right), \quad \varepsilon_a = 2 - \cos(k_a) - \cos(k'_a), \tag{49}$$

where  $k_a, k'_a$  are the momenta (40).

Note that

$$\Omega_1 \Omega_2 \cdots \Omega_p = \omega_1^M \omega_2^M \cdots \omega_p^M \tag{50}$$

(it follows from  $P_p[\{\omega, \omega^M\}] = \omega_1^M \cdots \omega_p^M$ , equation (43)), therefore the mass centre of  $p$ -particles' state moves accordingly to (19), (20).

Equations (46) provide real momenta  $k_a$  and  $k'_a$  in (40) if  $q$  is real,  $0 < q < 1$ . In addition, numerical estimations for not too big lattices and for two particles show that the ansatz is complete.

#### 4.4. Interpretation of the model

The one-dimensional limit of (46) must be mentioned firstly. In the case  $N = 1$ , our system becomes the chain of the length  $M - 1$ . Polynomial  $\mathfrak{P}(\omega|\{\Omega, \Omega\})$  has a simple structure, all  $\Omega_a$  may be excluded from (46). The resulting equations for  $\omega_a$  have the form of 'common' Bethe ansatz equations,

$$\omega_a^{M-1} = \prod_{b \neq a} \frac{G_{ab}(\{\omega\})}{G_{ba}(\{\omega\})} \equiv \prod_{b \neq a} \frac{q\omega_a - q^{-1}\omega_b}{q^{-1}\omega_a - q\omega_b}. \tag{51}$$

Let

$$q^2 = e^{-\hbar c}, \quad \omega_a = e^{i\hbar p_a}, \quad \hbar(M - 1) = L \tag{52}$$

and consider the limit  $\hbar \rightarrow 0$  with  $L$ -finite. Then (51) become the famous Bose gas equations [10]

$$e^{iL p_a} = \prod_{b \neq a} \frac{p_a - p_b + ic}{p_a - p_b - ic}. \tag{53}$$

Therefore, the term 'lattice Bose gas' is an apt one for equations (51) since we have the notion of quantum chain.

For generic  $N$  and  $M$ , it is appropriate to call the model 'two-dimensional lattice Bose gas'.

Since the notion of the lattice underlies the formulation of the system, the model is initially anisotropic. We are interesting in the thermodynamical limit of the system, when the size of the lattice  $N, M$  and the number of the particles  $p$  tend to infinity. The final conclusion about the thermodynamical isotropy/anisotropy is to be based on a form of density distribution  $\rho(k, k')$  of the momenta  $(k_a, k'_a)$  (40) for the ground state and on a structure of its excitations.

#### 4.5. Sketch derivation of (46)

Equations (46) may be obtained in the same way as (32). One may start with the case when  $m_1 < m_2 < \dots < m_p$  and  $m_p < m_1 + M$ . Then, the eigenstate of  $\tau_\alpha, \tau_\alpha |\Psi\rangle = |\Psi\rangle \tau_\alpha$ ,  $\tau_\alpha = \sum_a \omega_a^{-1}$ , is given by

$$\Psi_{m_1, \dots, m_p} = \sum_\sigma A_\sigma^{(0)} \omega_{\sigma_1}^{m_1} \omega_{\sigma_2}^{m_2} \dots \omega_{\sigma_p}^{m_p}. \quad (54)$$

The eigenvalue equation for  $\tau_\alpha$  provides the way to interpolate  $\Psi_{m_1, m_2, \dots, m_p}$  for the larger values of  $m_a$ . For instance, when  $m_p \sim m_1 + M$ , the eigenvalue equation modifies as

$$\tau_\alpha \Psi_{m_1, \dots, m_p} = q^{\delta_{m_p, m_1+M-1}} \Psi_{m_1-1, \dots, m_p} + q^{\delta_{m_p, m_1+M}} \Psi_{m_1, \dots, m_p-1} + \text{all the rest.} \quad (55)$$

Extra  $q$ -factors produce some linear transformation of the amplitudes  $A_\sigma$  of  $\Psi_{m_1, \dots, m_p} |_{m_p \geq m_1+M}$  in the same way as in equation (29). Repeating this procedure for  $m_p \sim m_2 + M, m_p \sim m_3 + M$ , etc, one comes to

$$\Psi_{m_1, m_2, \dots, m_p+M} = \sum_\sigma A_\sigma^{(1)} \omega_{\sigma_1}^{m_1} \omega_{\sigma_2}^{m_2} \dots \omega_{\sigma_p}^{m_p}, \quad (56)$$

where  $A_\sigma^{(1)}$  is a linear combination of  $A_\sigma^{(0)}$ :

$$A_\sigma^{(1)} = \sum_{\sigma'} \mathcal{M}_{\sigma, \sigma'} A_{\sigma'}^{(0)}, \quad (57)$$

cf (29). The ansatz (38) corresponds to (56) in the basis of eigenvectors of  $\mathcal{M}$ . The miracle of the exact integrability is that  $p! \times p!$  matrix  $\mathcal{M}$  has only  $p$  eigenvalues:

$$\det(\Omega - \mathcal{M}) = \mathfrak{F}(\Omega | \{\omega, \omega^M\})^{(p-1)!} \quad (58)$$

The second equation in (46), providing the  $N \leftrightarrow M$  symmetry of the Bethe ansatz equations, solves the boundary condition analogously to (31).

## 5. Conclusion

This paper presents a model of two-dimensional lattice Bose gas and the conjecture for the equations (46) describing its eigenstates. The model has the main features of a physical model. The states are described by the system of plane waves—at least the notion of the momenta is well defined. In particular, the sum of cosines of momenta is a candidate for the Hamiltonian. Equations (46), in the case when the sizes of lattice  $N, M$  and the number of bosons  $p$  are big, are the subject of further investigations. These equations entangle the sets of  $\omega_a$  and  $\Omega_a$  in a sophisticated way, so that the standard way of a root density derivation fails, and equations (46) need a development of a special technique. The limit (52) when  $q = e^{-\hbar c} \rightarrow 1$  with  $N' = \hbar N, M' = \hbar M$  and  $p' = \hbar p$  being finite does not simplify the equations.

We believe, this model may have some interest for the theory of integrable systems and for the condensed matter physics.

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